Modeling control systems with fuzzy differential equation

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Abstract. Fuzzy automatic control system synthesis is suggested for objects described with fuzzy differential equations with crisp coefficients. An illustrative numerical example solving fuzzy differential equation is presented.

Keywords:
Fuzzy differential equations , H-difference , Control System , Stability

1 Introduction

In engineering, dealing with uncertainties in control system design is a common problem in different branches of industry, this due to uncertainties that must to be added to an object model a priori or a posteriori.

Fuzzy modeling is a common way to consider that uncertainties, and this fuzzy models are based in Zadeh's [1–3] pioneer work. There exist wide literature [4–19] on fuzzy control, but these literature is based in Mamdani and Takagi-Sugeno type models, and do not consider a fuzzy differential equation modeling approach.

In this paper is presented an approach of modeling control systems with fuzzy differential equations.

2 Basic concepts and fuzzy differential equation

In this section, we give definitions which will used throughout the paper. See [20, 21]. Let A be a nonempty set. A fuzzy set u in A is characterized by its membership function $u : A \to [0, 1]$. Then $u(x)$ is interpreted as the degree of membership of a element x in the fuzzy set u for each $x \in A$.

Let $\mathbb{R}^n_F$ be the space of the all compact and convex fuzzy sets on $\mathbb{R}^n$.

Definition 1. *For* $u, v \in \mathbb{R}^n_F$, $(u \oplus v)(x) = \sup_{x_1+x_2=x} min \{u(x_1), v(x_2)\}$

The metric structure is given by the Hausdorff distance. [21]

Definition 2. *For $u \in \mathbb{R}^n_F$. The α-cut is the set* $[u]^\alpha = \{s \in \mathbb{R}^n : u(s) \geq \alpha\}$, $0 < \alpha < 1$.

Definition 3. *Let $u, v \in \mathbb{R}^n_F$. If there exists $w \in \mathbb{R}^n_F$ such that $u = v \oplus w$, then w is called the H-difference of u and v and it is denote by $u \ominus v$.*

Definition 4. *Let $F : T \to \mathbb{R}^n_F$ and $t_0 \in T$. The function F is said to be diferentiable at t_0 if:*
(I) an element $F'(t_0) \in \mathbb{R}^n_F$ exist such that, for all $h > 0$ sufficiently near 0, there are $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

are equal to $F'(t_0)$.
(or)
(II) there is an element $F'(t_0) \in \mathbb{R}^n_F$ exist such that, for all $h < 0$ sufficiently near 0, there are $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits

$$\lim_{h \to 0^-} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0^-} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

are equal to $F'(t_0)$.

Note that if F is differentiable in the first form (I), then it is not differentiable in the second form (II) and viceverse.

Theorem 1. *Let $F : T \to \mathbb{R}^n_F$ and $[F(t)]^\alpha = [F^\alpha_L(t), F^\alpha_R(t)]$, for each $\alpha \in [0, 1]$.Then (i) if F is differentiable in the fist form (I) then $F^\alpha_L(t), F^\alpha_R(t)$ are differentiable functions and*

$$[F'(t)]^\alpha = [(F^\alpha_L(t))', (F^\alpha_R(t))'], \tag{1}$$

(ii) if F is differentiable in the fist form (II) then $F^\alpha_L(t), F^\alpha_R(t)$ are differentiable functions and

$$[F'(t)]^\alpha = [(F^\alpha_R(t))', (F^\alpha_L(t))']. \tag{2}$$

Consider the fuzzy differential equation with crisp coefficients:

$$\tilde{X}^{(n)} + a_1 \tilde{X}^{(n-1)} + \ldots + a_n \tilde{X}(t) = k_{ob} \tau(t) \tag{3}$$

where a_i $i = 1 \ldots n$, and k_{ob} denote the constant coefficients that are crisp numbers. $\tilde{X}(t)$ denote an unknown fuzzy self-valued output variable. $\tau(t)$ is a fuzzy self-valued control action, t is time. $\tilde{X}^{(i)}$ are the i-th derivatives.

The fuzzy function $\tilde{X}(t)$ has the following properties:

$$(X(t))^\alpha = [X^\alpha_L(t), X^\alpha_R(t)] \tag{4}$$

$$\tilde{X}(t) = \cup_{\alpha\in(0,1]}\alpha X^{\alpha}(t); \forall\alpha \in (0,1] \tag{5}$$

$$(X^{\alpha}(t))^{i} = [(X_L^{\alpha}(t))^{(i)}, (X_R^{\alpha}(t))^{(i)}] \tag{6}$$

$$(\tilde{X}^{(i)}(t)) = \bigcup_{\alpha\in(0,1]} \alpha[(X_L^{\alpha}(t))^{(i)}, (X_R^{\alpha}(t))^{(i)}] \tag{7}$$

Let the control action $\tau(t)$ is formed subject to fuzzy functions $X^{(i)}(t).i = 1\ldots n$

$$\tau(t) = -\sum_{j=0}^{r} k_{p_j} X^{(j)}(t) \tag{8}$$

where k_{p_j} $j = 1\ldots r$ denote the parameters of controller tuning and are crisp numbers.

In most cases the stability degree is the main index of performance for fuzzy control systems [22] .Taking this into account, the problem of synthesis of control system dynamic object (3) can be formulated as follows.

It requires to determine such control (8) type, that transfers the dynamic object, described by fuzzy differential equation, from a given fuzzy initial state

$$\tilde{X}(t_0) = \tilde{X}_0, \tilde{X}^{(i)}(t_0) = \tilde{X}_0^{(i)}, (i = 1\ldots n-1) \tag{9}$$

to finite state

$$\tilde{X}(T) = \tilde{0} \tag{10}$$

providing maximal stability degree:

$$J = \max_{k_{pi}\in K_{pi}} \{-\mathrm{Re}\lambda_{\alpha}(a_1, a_2, \ldots a_n, k_{p1}, k_{p2}, \ldots k_{pm}\} \tag{11}$$

where

$$K_{pi} = [K_{pi}^{\min}, K_{p_i}^{\max}], K_{p_i}^{\min} \geq 0, i = 1\ldots r. \tag{12}$$

3 Synthesis of fuzzy control system

The formulated problem of synthesis of fuzzy control system (controller) (3), (8-12) of a high order can be solver on a base of the iterative approach [22]. That is, beginning from some $k_{p_i}^{0}$ for every parameter k_{p_i}, adding some definite $\triangle k_{p_i}$ and fixing transient processes in system (3), (8), that satisfy initial and finite conditions (9) and (10), the certain fuzzy solution sets are determined. Then we can choose an optimal solution from this set, i.e the optimal value k_{p_i}, that provides the maximal stability degree [22].

In practice many technological objects (including robotic manipulator as object of automatic control) are described by differential equations as a rule of the second or the third order. Taking into account the following, the formulated problem of parametric synthesis of control (3), (8-12) can be solved analytically by a method like suggested in [22].

Let an order of fuzzy differential equation, describing a control system be $n = 2$, the order of fuzzy controller (8) is $r = 0$, taking this into account the characteristic equations of fuzzy control system will be described by the following fuzzy differential equation:

$$\tilde{X}'' + a_1\tilde{X}' + (a_2 + k_{ob}k_{p0})\tilde{X}(t) = \tilde{0} \tag{13}$$

$$\tilde{X}(t_0) = \tilde{X}_0, \tilde{X}'(t_0) = \tilde{X}_0', \tag{14}$$

here 0 is a fuzzy zero. Then based on $[23, 24]$ we can write the following expressions for fuzzy differential equation (17), and also for initial conditions (14):

$$(X_L^\alpha)''(t) + a_1(X_L^\alpha)'(t) + (a_2 + k_{ob}k_{p0})X_L(t) = 0_L^\alpha \tag{15}$$

$$(X_R^\alpha)''(t) + a_1(X_R^\alpha)'(t) + (a_2 + k_{ob}k_{p0})X_R(t) = 0_R^\alpha \tag{16}$$

Is should be noted that an interval $0^\alpha = [0_L^\alpha, 0_R^\alpha]$, which is and α-cut of the fuzzy zero, is sufficiently small and, in particular case, can be singleton.

From the convergence condition of solution of differential equations, or providing of control system stability and maximal performance criterion, stability degree of the tuning parameter k_{p_0} of fuzzy controller (8) can be determined in accordance with [22].

4 Damping analysis

In examples consider fuzzy differential equation

$$\tilde{X}'' + a_1\tilde{X}' + (a_2 + k_{ob}k_{p0})\tilde{X}(t) = \tilde{0} \tag{17}$$

with $X' = [-0.001, 0.001]$, $\tilde{0} = [-0.0001, 0.0001]$ and $X_0 = [2, 4]$.

4.1 Overdamped case

With $a_1 = 4.5$, $a_2 = 0.95$, $k_{ob} = 20$, $k_{p0} = 1.7$, then we obtain

$$(X_L^\alpha)''(t) + 4.5(X_L^\alpha)'(t) + 4.35X_L(t) = -0.001, X_L = 2, X' = -0.0001 \tag{18}$$

$$(X_R^\alpha)''(t) + 4.5(X_R^\alpha)'(t) + 4.35X_L(R) = 0.001, X_R = 4, X' = 0.0001 \tag{19}$$

and solving (18-19), results:

$$X_L(t) = 3.66551e^{-1.4059028t} - 1.66551e^{-3.0940972t} - 0.0000229885 \tag{20}$$

and

$$X_R(t) = 7.33078e^{-1.4059028t} - 3.33101e^{-3.0940972t} + 0.0000229885, \tag{21}$$

which response is depicted in Fig. 1.

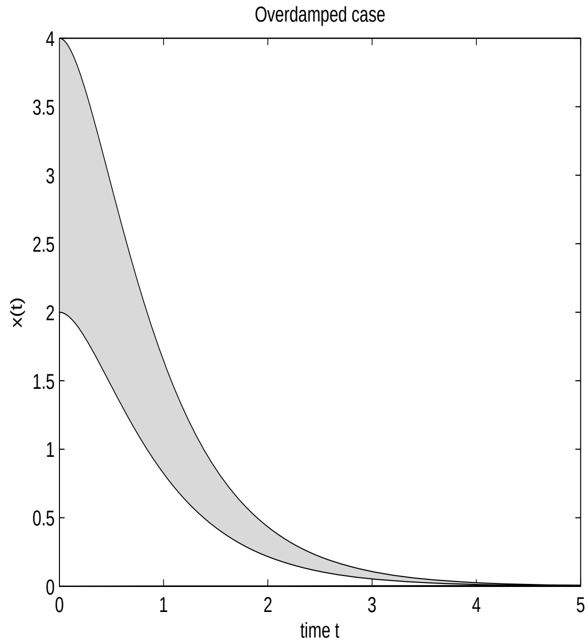

Fig. 1. Over damped case

4.2 Critically damped case

With $a_1 = 4.5$, $a_2 = 1$, $k_{ob} = 20$, $k_{p0} = 1.5$, then we obtain

$$(X_L^\alpha)''(t) + 4(X_L^\alpha)'(t) + 4X_L(t) = -0.001, X_L = 2, X' = -0.0001 \qquad (22)$$

$$(X_R^\alpha)''(t) + 4(X_R^\alpha)'(t) + 4X_R(t) = 0.001, X_R = 4, X' = 0.0001 \qquad (23)$$

and solving (22-23), results:

$$X_L(t) = 2.00025e^{-2t} + 4.0004te^{-2t} - 0.00025 \qquad (24)$$

and

$$X_R(t) = 3.999755e^{-2t} + 7.9996te^{-2t} + 0.00025, \qquad (25)$$

which response is depicted in Fig. 2.

4.3 Underdamped case

With $a_1 = 2.4$, $a_2 = 0.95$, $k_{ob} = 20$, $k_{p0} = 1.7$, then we obtain

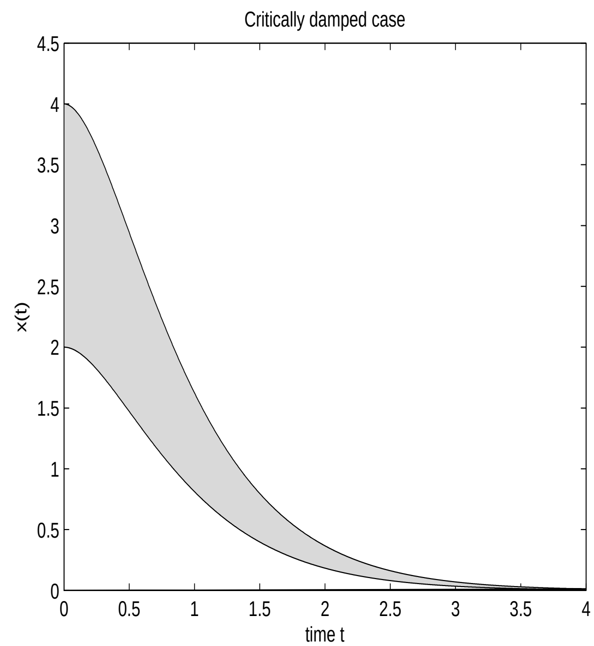

Fig. 2. Critically damped case.

$$(X_L^{\alpha})''(t) + 2.4(X_L^{\alpha})'(t) + 4.35X_L(t) = -0.001, X_L = 2, X' = -0.0001 \tag{26}$$

$$(X_R^{\alpha})''(t) + 2.4(X_R^{\alpha})'(t) + 4.35X_R(t) = 0.001, X_R = 4, X' = 0.0001 \tag{27}$$

and solving (26-27), results:

$$X_L(t) = 2.00023e^{-1.2t}\cos(1.7058722t) + 1.40701e^{-1.2t}\sin(1.7058722t) - 0.000229885 \tag{28}$$

and

$$X_R(t) = 2.81371e^{-1.2t}\cos(1.7058722t) + 3.99977e^{-1.2t}\sin(1.7058722t) + 0.000229885, \tag{29}$$

which response is depicted in Fig. 3.

5 The case study

Consider fuzzy differential equation

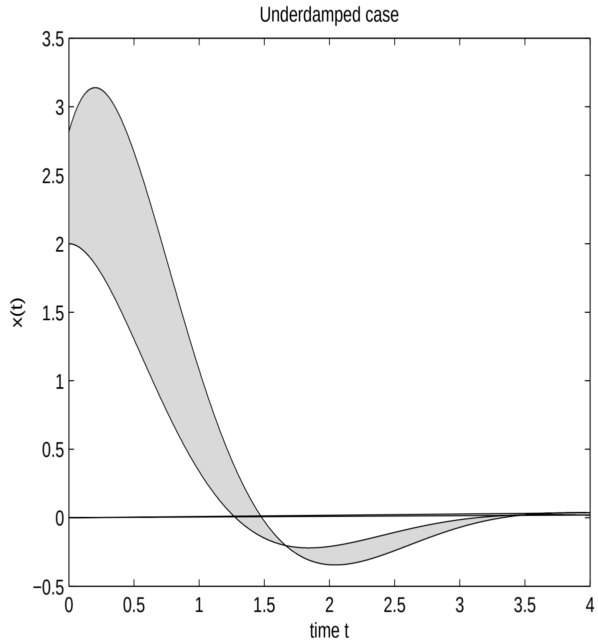

Fig. 3. Underdamped case.

$$\tilde{X}'' + a_1\tilde{X}' + a_2\tilde{X}(t) = \tilde{0} \tag{30}$$

with $X' = [-0.001, 0.001]$, $\tilde{0} = [-0.0001, 0.0001]$, using (15)-(16),

$$(X_L^\alpha)''(t) + a_1(X_L^\alpha)'(t) + a_2X_L(t) = 0_L^\alpha \tag{31}$$

$$(X_R^\alpha)''(t) + a_1(X_R^\alpha)'(t) + a_2X_R(t) = 0_R^\alpha \tag{32}$$

is obtained.

The motion is determined by the roots of the characteristic equation of (30).
If the roots are real and unequal, the motion is overdamped, if the roots are real and equal, the motion is critically damped, finally, if the roots are conjugate complex numbers, the motion is underdamped.

In Fig. 4 , a small block of mass m is attached to one end of a spring, and the other end of the unstretched spring is attached to a fixed wall. Assuming that the block is acted on by a force of friction that opposes the motion.

If the block weigh is $32lb$, the spring constant $k = 36lb/ft$, and the resistance coefficient is $b = 13$, the resulting system equation is:

$$\tilde{X}'' + 13\tilde{X}' + 36\tilde{X}(t) = \tilde{0} \tag{33}$$

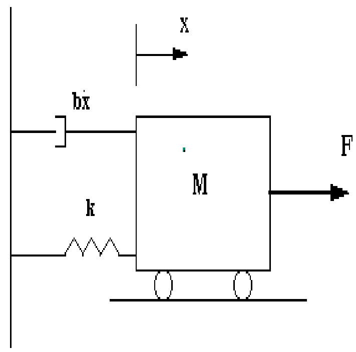

Fig. 4. mass-damper-spring

with $X' = [-0.001, 0.001]$, $\tilde{0} = [-0.0001, 0.0001]$ and $X_0 = [1.9, 2.1]$, and solving (33), results on a overdamped motion because

$$X_L(t) = 3.7802e^{-4t} - 1.6802e^{-9t} \tag{34}$$

and

$$X_R(t) = 3.4198e^{-4t} - 1.5198e^{-9t} \tag{35}$$

and the system's response is depicted in Fig. 5.

6 Conclusions

The approach reported in this paper helps in the modeling of uncertainties directly in fuzzy differential equations, and the reported cases of study gives an approximation that extends control engineering theory to the fuzzy case according with the need of dealing with those uncertainties.

The resulting family of solutions to the fuzzy differential equations gives a footprint of uncertainty, like the defined in [25], which helps to conclude that the results reported in this paper are according with the type-2 fuzzy [25] logic theory of the computing with words paradigm. Finally, the formulas for computing second order fuzzy differential equation was obtained.

References

1. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning - i. Inf. Sci. **8**(3) (1975) 199–249
2. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning - ii. Inf. Sci. **8**(4) (1975) 301–357
3. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning - ii. Inf. Sci. **9**(1) (1975) 43–80

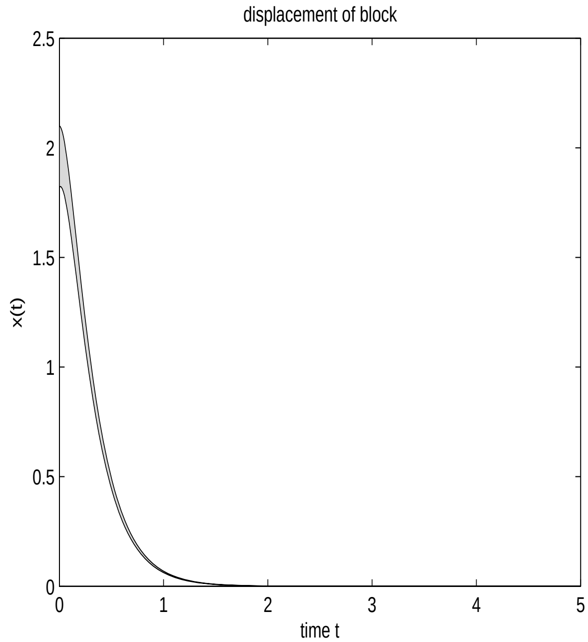

Fig. 5. Block displacement

4. Lim, C., Hiyama, T.: Application of fuzzy logic control to a manipulator. Robotics and Automation, IEEE Transactions on **7**(5) (oct 1991) 688 –691
5. Newton, R., Xu, Y.: Neural network control of a space manipulator. Control Systems, IEEE **13**(6) (dec. 1993) 14 –22
6. Llama, M., Kelly, R., Santibañez, V.: Stable computed-torque control of robot manipulators via fuzzy self-tuning. Systems, Man, and Cybernetics, Part B: Cybernetics, IEEE Transactions on **30**(1) (feb 2000) 143 –150
7. Yoo, B.K., Ham, W.C.: Adaptive control of robot manipulator using fuzzy compensator. Fuzzy Systems, IEEE Transactions on **8**(2) (apr 2000) 186 –199
8. Tsai, C.H., Wang, C.H., Lin, W.S.: Robust fuzzy model-following control of robot manipulators. Fuzzy Systems, IEEE Transactions on **8**(4) (aug 2000) 462 –469
9. Peng, L., Woo, P.Y.: Neural-fuzzy control system for robotic manipulators. Control Systems, IEEE **22**(1) (feb 2002) 53 –63
10. Sun, Y.L., Er, M.J.: Hybrid fuzzy control of robotics systems. Fuzzy Systems, IEEE Transactions on **12**(6) (dec. 2004) 755 – 765
11. Santibañez, V., Kelly, R., Llama, M.: Global asymptotic stability of a tracking sectorial fuzzy controller for robot manipulators. Systems, Man, and Cybernetics, Part B: Cybernetics, IEEE Transactions on **34**(1) (feb. 2004) 710 – 718
12. Santibañez, V., Kelly, R., Llama, M.: A novel global asymptotic stable set-point fuzzy controller with bounded torques for robot manipulators. Fuzzy Systems, IEEE Transactions on **13**(3) (june 2005) 362 – 372

13. Merchan-Cruz, E., Morris, A.: Fuzzy-ga-based trajectory planner for robot manipulators sharing a common workspace. Robotics, IEEE Transactions on **22**(4) (aug. 2006) 613 –624
14. Chen, C.S.: Supervisory adaptive tracking control of robot manipulators using interval type-2 tsk fuzzy logic system. Control Theory Applications, IET **5**(15) (13 2011) 1796 –1807
15. Meza, J.L., Santibanez, V., Soto, R., Llama, M.A.: Fuzzy self-tuning pid semiglobal regulator for robot manipulators. Industrial Electronics, IEEE Transactions on **59**(6) (june 2012) 2709 –2717
16. Castillo, O., Aguilar, L., Cazarez, N., Cardenas, S.: Systematic design of a stable type-2 fuzzy logic controller. Applied Soft Computing **8**(3) (2008) 1274 – 1279.
17. Cazarez-Castro, N., Aguilar, L., Castillo, O., Rodriguez, A.: Optimizing type-1 and type-2 fuzzy logic systems with genetic algorithms. Research in Computing Science **39** (October 2008) 131–153
18. Cazarez-Castro, N.R., Aguilar, L.T., Castillo, O.: Fuzzy logic control with genetic membership function parameters optimization for the output regulation of a servomechanism with nonlinear backlash. Expert Systems with Applications **37**(6) (2010) 4368 – 4378
19. Cazarez-Castro, N.R., Aguilar, L.T., Castillo, O.: Designing type-1 and type-2 fuzzy logic controllers via fuzzy lyapunov synthesis for nonsmooth mechanical systems. Engineering Applications of Artificial Intelligence **25**(5) (2012) 971 – 979
20. Puri, M.L., Ralescu, D.A.: Differentials of fuzzy functions. Journal of Mathematical Analysis and Applications **91**(2) (1983) 552 – 558
21. Kaleva, O.: Fuzzy differential equations. Fuzzy Sets and Systems **24**(3) (1987) 301 – 317
22. Zeynalov E. R., Jafarov P. S., J.S.M.: Synthesis of fuzzy controllers for astatic objects, described by differential equations with fuzzy coefficients, Antalya, Turkey (2007) 290 – 295
23. Kaleva, O.: A note on fuzzy differential equations. Nonlinear Analysis: Theory, Methods and Applications **64**(5) (2006) 895 – 900
24. Li, Z.: Fuzzy Chaotic Systems. Springer (2006)
25. Mendel, J.: Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions. Prentice Hall, Upper Saddle River, NJ (2001)